# THE MATHEMATICS STUDENT

Volume IX] SEPTEMBER 1941 [Number 3

## ON JAINA MAGIC SQUARES

BY

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This note contains a mnemonic for the construction of Jaina magic squares of order four. The mnemonic is given later.

A magic square of order $n$ is a square of $n^2$ cells in which the numbers 1, 2, 3, . . . $n^2$ are so arranged that the sum of the numbers in each row and in each column is the same. This sum is easily seen to be 34 if $n=4$. The number of magic squares of order four was calculated by D. N. Lehmer* and was found to be 549,504; he also found that less than 1½ percent—actually, 7040—of these magic squares are such that the sum of the numbers in each of the two main diagonals too is 34. Less than one in a thousand magic squares of order four have the Jaina property of which an account will be given later; the number of such squares is 384.

Certain magic squares of order four are known as Jaina† squares. They are also known as Nasik,§ perfect diabolic and pan-diagonal §§ squares. This type of magic square is "the most perfect conceivable", and is very much more regular than any other type. "It represents a distribution of figure values in what might be called absolute equilibrium".

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* *Bulletin of the American Math. Soc.*, 39 (1933) 764-767

" " 40 (1934) 496

" " 39 (1933) 981

† In 1904 Professor Schilling communicated to the Mathematical Society of Göttingen the fact that Professor Kielhorn had found a Jaina inscription in the city of Khajuraho and that the inscription included a magic square of notable peculiarity that each subsquare sums to 34. The inscription is probably of the eleventh century.

§ They were named as Nasik squares by Rev. Mr. A. H. Frost who lived at Nasik. It will be more appropriate to call these squares Indian Squares, but the other names have become fairly established.

§§ This term is explained later. For the general theory of pan-diagonal squares of order four, see J. B. Rosser and R. J. Walker, *Bulletin of the American Math. Soc.*, 44 (1938) 416-420; such properties of Jaina squares as are mentioned in the present note have either been proved in the above paper, or can be deduced from the results there in, or can be proved similarly.

Books with titles such as *mathematical curiosities*, or *mathematical recreations*, or *amusements in mathematics* are sure to give some account of magic squares. It is probably true that magic squares are popular in most countries. The author of this note learnt by heart at the age of nine the following pan-diagonal square which was taught to him by an elderly person who had not been to school at all.

| | | | |
|---|---|---|---|
| 8 | 11 | 2 | 13 |
| 1 | 14 | 7 | 12 |
| 15 | 4 | 9 | 6 |
| 10 | 5 | 16 | 3 |

(X)

A magic square of order four is said to be pan-diagonal if the integers 1, 2, 3, . . . 16 are so arranged in it that the sum of the numbers in each of the four rows, four columns and eight diagonals is 34. There are eight diagonals because we take into account the broken diagonals too; there are six broken diagonals and the numbers belonging to two such in X are 15, 5, 2, 12 and 15, 14, 2, 3. If we repeatedly write the magic square (X) in the manner shown below in (Y) it is seen that any four consecutive numbers taken along a diagonal direction give the numbers that go to form a diagonal (broken or otherwise) of (X).

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 8 | 11 | 2 | 13 | 8 | 11 | 2 | 13 |
| 1 | 14 | 7 | 12 | 1 | 14 | 7 | 12 |
| 15 | 4 | 9 | 6 | 15 | 4 | 9 | 6 |
| 10 | 5 | 16 | 3 | 10 | 5 | 16 | 3 |
| 8 | 11 | 2 | 13 | 8 | 11 | 2 | 13 |
| 1 | 14 | 7 | 12 | 1 | 14 | 7 | 12 |
| 15 | 4 | 9 | 6 | 15 | 4 | 9 | 6 |
| 10 | 5 | 16 | 3 | 10 | 5 | 16 | 3 |

(Y)

Every pan-diagonal magic square of order four has the following properties as well:

(1) The sum of the four numbers in each square of order two is again 34. There are sixteen such distinct squares as is easily seen from (Y).

(2) The sum of the four numbers at the four corners of each square of order three* is 34. A more precise result is that the sum of any two alternate numbers in any diagonal line is 17.

(3) If in a pan-diagonal magic square of order four, $q$, $r$, $s$, $t$ are adjacent† to $p$ then the same four numbers are adjacent to $p$ in all the 384 pan-diagonal squares.

Since 34 is the sum of the numbers in each row, column, or square of order two it is clear that a pan-diagonal magic square of order four becomes determinate when we know the numbers $p$, $q$, $r$, $s$, $t$ occupying respectively the cells $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$ noted below.§

* The sum of the four numbers at the four corners of a pan-diagonal magic square is 34. As is seen from (Y) the numbers at the four corners of (X) are merely the numbers of the square of order two whose left hand top number is the right hand bottom number of (X).

† A number $g$ is adjacent to $h$ if $g$ and $h$ are in consecutive cells in a row or column we interpret suitably the word *consecutive* and note that every number $h$ ($h=1, 2, \ldots 16$) has four and only four adjacent numbers. For instance, in (X) 11, 7, 4 and 1 are adjacent to 14, and, as is more easily seen from (Y), the numbers 11, 1, 13 and 10 are adjacent to 8.

§ Indeed a pan-diagonal square becomes determinate when any three of the four numbers $q$, $r$, $s$, $t$ are known, for there is only one number $p$ that can be adjacent to three numbers in three of the four cells $\beta$, $\gamma$, $\delta$, $\varepsilon$. When four of the five numbers $p$, $q$, $r$, $s$, $t$ are known the fifth can be determined from the equation $2p+q+r+s+t=51$. Incidentally, it may be noted that there are two pan-diagonal squares, (Z) and (W) in which the eight numbers 1, 3, 5, 7, 16, 14, 12, 10 occupy the same positions as in (E)

| | | | |
|---|---|---|---|
| 1 | | | 14 |
| | 10 | 5 | |
| | 3 | 16 | |
| 12 | | | 7 |

(E)

| | | | |
|---|---|---|---|
| 1 | 8 | 11 | 14 |
| 15 | 10 | 5 | 4 |
| 6 | 3 | 16 | 9 |
| 12 | 13 | 2 | 7 |

(Z)

| | | | |
|---|---|---|---|
| 1 | 15 | 4 | 14 |
| 8 | 10 | 5 | 11 |
| 13 | 3 | 16 | 2 |
| 12 | 6 | 9 | 7 |

(W)

| | | | |
|---|---|---|---|
| | $\varepsilon$ | | |
| $\delta$ | $\alpha$ | $\beta$ | |
| | $\gamma$ | | |
| | | | |

The mnemonic given here is based on the above observation. We start from the numbers 3, 6, 9, 12, 15 and note that 3, 9, 12 and 15 are adjacent to 6. So, we put the number 6 in $\alpha$ (the second cell of the second row) and place the numbers 3, 9, 12, 15 in the adjacent cells $\beta$, $\gamma$, $\delta$, $\varepsilon$. This can be done in $4 \cdot 3 \cdot 2 = 24$ ways, and we can choose any one of them. Some of the stages in the construction of one such square is indicated below.

| | | | |
|---|---|---|---|
| | 3 | | |
| 12 | 6 | 9 | |
| | 15 | | |
| | | | |

| | | | |
|---|---|---|---|
| 13 | 3 | | |
| 12 | 6 | 9 | |
| | 15 | | |
| | | | |

| | | | |
|---|---|---|---|
| 13 | 3 | 16 | |
| 12 | 6 | 9 | |
| 1 | 15 | 4 | |
| | | | |

| | | | |
|---|---|---|---|
| 13 | 3 | 16 | 2 |
| 12 | 6 | 9 | 7 |
| 1 | 15 | 4 | 14 |
| 8 | 10 | 5 | 11 |

As a pan-diagonal magic square continues to be such after any cyclic permutation of its rows and columns it suffices to construct all such squares with a given number $g$ occupying a fixed cell $c$. Plainly there are $384/16 = 24$ such squares, and the mnemonic given above enables us to write down all the 24 squares in which 6 occupies the second cell in the second row.

A well known, popular and more general problem arising out of magic squares of order four is to choose and arrange sixteen distinct positive integers in a square of order four in such a way that the sum of the numbers in each row and in each column is a given number $T \geqslant 34$. One construction for such a square is to start from the (X) mentioned earlier and replace the numbers 13, 14, 15, 16 in it by the numbers $13+T-34$, $14+T-34$, $15+T-34$, $16+T-34$ respectively. Thus if $T=35$ We have the square

| | | | |
|---|---|---|---|
| 8 | 11 | 2 | 14 |
| 1 | 15 | 7 | 12 |
| 16 | 4 | 9 | 6 |
| 10 | 5 | 17 | 3 |

The square so obtained is such that $T$ is the sum of four numbers in each of (1) four rows, (2) four columns and (3) four diagonals, including the two main diagonals. The square is not, however, pan-diagonal.

A pan-diagonal square of this kind also possesses properties similar to those of the Jaina square and, in particular, is such that the sum of any two alternate numbers in any diagonal line is equal to $T/2$.* Hence the problem of constructing a pan-diagonal square for an odd number $T$ is insoluble. It is always possible to construct a pan-diagonal square if $T$ is even. One construction is to start from any one of the 384 pan-diagonal squares in which the integers 1, 2, . . . 16 are employed and replace the numbers 9, 10, . . . 16 by $9+\frac{1}{2}(T-34)$, $10+\frac{1}{2}(T-34)$, . . . . $16+\frac{1}{2}(T-34)$ respectively. For example, starting from (X) we get the following square when $T=36$

| | | | |
|---|---|---|---|
| 8 | 12 | 2 | 14 |
| 1 | 15 | 7 | 13 |
| 16 | 4 | 10 | 6 |
| 11 | 5 | 17 | 3 |

A more general construction is as follows: Let $S=\frac{1}{2}T$. Choose numbers $a$, $b$, $P$, $R$, such that $a_1$, $a_2$, . . . $a_{16}$ are distinct positive integers where

$$a_1=a,\ a_2=b,\ a_3=P-a,\ a_4=P-b,$$

$a_5$, $a_6$, $a_7$, $a_8$ are the numbers $R-a_1$, $R-a_2$, $R-a_3$, $R-a_4$ respectively,

and $a_9$, $a_{10}$, . . . $a_{16}$ are the numbers $S-a_1$, $S-a_2$, . . . $S-a_8$ respectively. The arrangement of these numbers in a pan-diagonal square is shown below in two stages.

| | | | |
|---|---|---|---|
| $a$ | $R-a$ | | |
| | | $b$ | $R-b$ |
| $P-a$ | $R-(P-a)$ | | |
| | | $P-b$ | $R-(P-b)$ |

| | | | |
|---|---|---|---|
| $a$ | $R-a$ | $S-(P-a)$ | $S-\{R-(P-a)\}$ |
| $S-(P-b)$ | $S-\{R-(P-b)\}$ | $b$ | $R-b$ |
| $P-a$ | $R-(P-a)$ | $S-a$ | $S-(R-a)$ |
| $S-b$ | $S-(R-b)$ | $P-b$ | $R-(P-b)$ |

* The proof given in the paper by J. B. Rossen and R. J. Walken will yield this conclusion.

The arrangement is effected in conformity with the results that (1) the sum of any two adjacent numbers in any row (column) is equal to the sum of the corresponding numbers in the alternate row (column), and, (2) the sum of any two alternate numbers in any diagonal line is $S$. It is easily seen that the above construction is a general one in the sense that every pan-diagonal square of order four can be obtained by a suitable choice of $a$, $b$, $P$, $R$.

It is curious to note that there are squares in which the same number $T$ is the sum of the numbers in each of the subsquares of order two, and in each of the eight diagonals, and yet the sums of the numbers in the columns are different. In the square given below the sum is 81 for each row and each diagonal, and therefore, for each subsquare too, but the sum of the number in no column is 81.

| | | | |
|---|---|---|---|
| 1 | 8 | 37 | 35 |
| 39 | 33 | 3 | 6 |
| 4 | 5 | 40 | 32 |
| 38 | 34 | 2 | 7 |

# ON THE EXTREME VALUES OF SAMPLES TAKEN FROM A RECTANGULAR POPULATION

BY

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Prof. Karl Pearson in a paper in *Biometrika* Vol. 23, [1], derived an expression for the $m$th moment of the range between $q$th and $q'$th ranked individuals, for samples of size $n$ taken from a rectangular population. The $m$th moment of the distance from origin, of the character in the $q$th rank, can be obtained by having $q=0$ and $q'=q$, in the expression derived by Karl Pearson. It is sometimes necessary to have expressions for the $m$th moments of the characters in the first and the last ranks of samples of size $n$ about their respective arithmetic means. In this paper expressions are derived for the $m$th moments, about the arithmetic mean, of the maximum and minimum values of samples of size $n$ taken from a rectangular population.

Prof. Pearson, by using the first four moments, of the range, about the origin obtained the equation for the curve* giving the distribution of the interval $x$ between individuals in the $q$th and $q'$th ranks when we take N samples of size $n$ from the population, where the mode is the origin and the range of the original rectangular population $\omega$. The frequency distributions of the maximum and minimum values of the characters of samples can be found by putting, $q=0$, $q'=n$ and $q=0$, $q'=1$ in that equation, to be

$$y = \frac{N}{\omega} \cdot n\ (1+x/\omega)^{n-1} \quad \ldots \quad (a)$$

$$y = \frac{N}{\omega} \cdot n\ (1-x/\omega)^{n-1} \quad \ldots \quad (b)$$

respectively. If $y=f(x)$ be the law governing any distribution where $x$ is measured from origin, and $\xi$ the deviation of the ranked character from its arithmetic mean, then $\mu_m$ the $m$th moment of the character of the ranked individual can be obtained by,

$$\mu_m = \int \xi^m\ y d\xi / \int y d\xi$$

the integration being extended throughout the range of the original population under consideration.

* *Biometrika.* Vol. 23. p. 373.

The equation ($a$), giving the distribution of the maximum values has the mode at $(\omega, 0)$ which is the origin. So shifting the origin to (0, 0) from $(\omega, 0)$, the equation reduces to $y=(x/\omega)^{n-1}$.

If $\bar{x}_n$ be the arithmetic mean of the maximum values we have,

$$\bar{x}_n \cdot \int_0^{\omega} (x/\omega)^{n-1}\, dx = \int_0^{\omega} (x/\omega)^{n-1}\, x\, dx$$

Using $x=\omega y$, this reduces to,

$$\bar{x}_n \cdot \int_0^{1} y^{n-1}\, dy = \int_0^{1} \omega .\, y^n\, dy$$

$$\text{or } \bar{x}_n = \frac{n}{(n+1)} \cdot \omega \qquad \ldots \quad (1)$$

Also, $\mu_m$, the $m$th moment of the maximun values about $\bar{x}_n$ becomes,

$$\mu_m \cdot \int_0^{\omega} (x/\omega)^{n-1}\, dx = \int_0^{\omega} \left(\frac{x}{\omega}\right)^{n-1} (x-\bar{x}_n)^m\, dx.$$

Using the same substitution $x=\omega y$, this reduces to

$$\mu_m \cdot \int_0^{1} y^{n-1}\, dy = \omega^m \int_0^{1} y^{n-1} \left(y - \frac{n}{n+1}\right)^m dy$$

$$\text{or } \mu_m = n \cdot \omega^m \int_0^{1} y^{n-1} \left(y - \frac{n}{n+1}\right)^m dy \qquad \ldots \quad (2)$$

$$= n \cdot \omega^m \int_0^{1} y^{n-1} \left[\frac{1}{n+1} - (1-y)\right]^m dy$$

Since $|y|<1$ always, expanding the second term in the integral we have,

$$\mu_m = n \cdot \omega^m \int_0^{1} y^{n-1}\, dy \left[\frac{1}{(n+1)^m} - \frac{m}{(n+1)^{m-1}} \cdot (1-y) + \frac{m(m-1)}{2!\,(n+1)^{m-2}} (1-y)^2 - \frac{m(m-1)(m-2)}{3!\,(n+1)^{m-3}} (1-y)^3 + \cdots \right]$$

$$= n \cdot \left(\frac{\omega}{n+1}\right)^m \cdot \int_0^{1} \left[y^{n-1} - m(n+1)\, y^{n-1} (1-y) + \frac{m(m-1)}{2!} \cdot (n+1)^2 \cdot y^{n-1} (1-y)^2 - \frac{m(m-1)(m-2)}{3!} \cdot (n+1)^3 \cdot y^{n-1} (1-y)^3 + \cdots \right] dy,$$

Using the values of the complete $\beta$-function, this reduces to

$$= n \cdot \left(\frac{\omega}{n+1}\right)^m \left[\frac{1}{n} - m(n+1) \cdot \frac{(n-1)!}{(n+1)!} + \frac{m(m-1)(n+1)^2}{2!} \cdot \frac{2!\,(n-1)!}{(n+2)!} - \frac{m(m-1)(m-2) \cdot (n+1)^3}{3!} \cdot \frac{3!\,(n-1)!}{(n+3)!} + \cdots \right]$$

$$= \left(\frac{\omega}{n+1}\right)^m \left[1 - \frac{m}{(n+1)} \cdot (n+1) + \frac{m(m-1)}{(n+1)(n+2)} (n+1)^2 - \frac{m(m-1)(m-2)}{(n+1)(n+2)(n+3)} (n+1)^3 + \cdots \right]$$

$$\mu_m = \left(\frac{\omega}{n+1}\right)^m \cdot F(-m, 1; n+1; n+1) \qquad \ldots \quad (3)$$

where F $(a, b; c; z)$ is the standard hypergeometric series,

$$1+\frac{a\cdot b}{1\cdot c}\cdot z+\frac{a(a+1)\cdot b(b+1)}{1\cdot 2\cdot c(c+1)}\cdot z^2+\cdots\cdots$$

Again if $\bar{x}_1$ be the arithmetic mean of the minimum values of samples, using the equation $(b)$, we have

$$\bar{x}_1\cdot\int_0^{\omega}(1-x/\omega)^{n-1}\,dx=\int_0^{\omega}(1-x/\omega)^{n-1}\,x\,dx,$$

Substituting $x=\omega y$, this reduces to

$$\bar{x}_1\cdot\int_0^{1}(1-y)^{n-1}\,dy=\int_0^{1}\omega\cdot(1-y)^{n-1}\,y\,dy$$

$$\text{or}\quad \bar{x}_1=\frac{1}{(n+1)}\cdot\omega \qquad \ldots \quad (4)$$

Also, $\nu_m$ the $m$th moment of the minimum values about $\bar{x}_1$ reduces to,

$$\nu_m\cdot\int_0^{\omega}(1-x/\omega)^{n-1}\,dx=\int_0^{\omega}(1-x/\omega)^{n-1}(x-\bar{x}_1)^m\,dx.$$

With the usual transformation, $x=\omega y$, we have

$$\nu_m\cdot\int_0^{1}(1-y)^{n-1}\,dy=\omega^m\cdot\int_0^{1}(1-y)^{n-1}\left(y-\frac{1}{n+1}\right)^m dy$$

$$\text{or}\quad \nu_m=n\cdot\omega^m\int_0^{1}(1-y)^{n-1}\left(y-\frac{1}{n+1}\right)^m dy \qquad \ldots \quad (5)$$

since $|y|<1$. Expanding the second term in the integral,

$$\nu_m=n\cdot(-1)^m\cdot\omega^m\cdot\int_0^{1}(1-y)^{n-1}\,dy\left[\frac{1}{(n+1)^m}-\frac{m}{(n+1)^{m-1}}\,y+\frac{m(m-1)}{2!\,(n+1)^{m-2}}\cdot y^2\cdot-\frac{m(m-1)(m-2)}{3!\,(n+1)^{m-3}}\cdot y^3+\cdots\right]$$

$$=n(-1)^m\cdot\left(\frac{w}{n+1}\right)^m\cdot\int_0^{1}\left[(1-y)^{n-1}-m(n+1)(1-y)^{n-1}y+\frac{m(m-1)}{2!}(n+1)^2(1-y)^{n-1}y^2-\frac{m(m-1)(m-2)}{3!}\cdot(n+1)^3(1-y)^{n-1}y^3+\cdots\right]dy$$

Using the values of complete $\beta$-function, this reduces to

$$n(-1)^m\cdot\left(\frac{\omega}{n+1}\right)^m\left[\frac{1}{n}-m(n+1)\cdot\frac{(n-1)!}{(n+1)!}+\frac{m(m-1)(n+1)^2}{2!}\cdot\frac{2!\,(n-1)}{(n+2)!}-\cdots\right]$$

$$=(-1)^m\left(\frac{\omega}{n+1}\right)^m\left[1-\frac{m}{(n+1)}\cdot(n+1)+\frac{m(m-1)}{(n+1)(n+2)}(n+1)^2-\frac{m(m-1)(m-2)}{(n+1)(n+2)(n+3)}(n+1)^3+\cdots\right]$$

$$\nu_m=(-1)^m\left(\frac{\omega}{n+1}\right)^m\cdot F(-m,\,1;\,n+1;\,n+1) \qquad \ldots \quad (6)$$

These expressions for the $m$th moments of the extreme values can also be derived independently without using the equation of the frequency distribution obtained by Prof. Karl Pearson. In the following the $m$th moments of the extreme values of samples about their arithmetic means, are obtained by using a method similar to that adopted by Mr. N. Sundararama Sastry in deriving an expression [2] for the $m$th moment of the range about the arithmetic mean of samples taken from a rectangular population.

As 0 to $\omega$ represents the range of the original rectangular population from which the samples of size $n$ are taken, all the values of the variable between 0 and $\omega$ are of equal frequency of occurrence in the population.

Let $x_1$ be the character of the first individual, and $x_n$ that of the last in a sample of size $n$ and the intermediate values are $x_2, x_3 \cdot \cdot \cdot \cdot \cdot x_{n-1}$ such that

$$x_1 \leqslant x_2 \leqslant x_3 \cdot \cdot \cdot \cdot \cdot \leqslant x_n.$$

Then, assuming that the original population is infinite, the chance that we have one individual at $x_n \pm \frac{1}{2} dx_n$ and the remaining $(n-1)$ between 0 and $x_n$ is

$$p_1 = \frac{n\,!}{(n-1)\,!}\left(\frac{dx_n}{\omega}\right)\left(\frac{x_n}{\omega}\right)^{n-1}$$

$$= \frac{n}{\omega^n}\, x_n^{n-1}\, dx_n \cdot$$

$x_n$ of the sample may be any where between 0 to $\omega$. Then $\bar{x}_n$ the mean of the maximum values of samples, reduces to

$$\bar{x}_n = \int_0^\omega p_1 \cdot x_n = \int_0^\omega \frac{n}{\omega^n} \cdot x_n^n\, dx_n = \frac{n}{(n+1)} \cdot \omega.$$

which is same as the result (1) already obtained.

Again, $\mu_m$, the $m$th moment becomes,

$$\mu_m = \int_0^\omega p_1\, (x_n - \bar{x}_n)^m$$

$$= \int_0^\omega \frac{n}{\omega^n} \cdot x_n^{n-1} \left(x_n - \frac{n}{n+1} \cdot \omega\right)^m dx_n \cdot$$

Using the substitution, $x_n = \omega y$ we have,

$$\mu_m = n\, \omega^m \int_0^1 y^{n-1} \left(y - \frac{n}{n+1}\right)^m dy \qquad \dots \quad (7)$$

which is the same as the equation (2). So, by integrating we arrive at the same hypergeometric series (3) as the value for $\mu_m$.

Similarly, the chance that we have one individual at $x_1 \pm \frac{1}{2} dx$, and the remaining $(n-1)$ between $x_1$ and $\omega$, is

$$p_2 = \frac{n!}{(n-1)!} \left(\frac{dx_1}{\omega}\right)\left(\frac{\omega - x_1}{\omega}\right)^{n-1}$$

$$= \frac{n}{\omega^n} (\omega - x_1)^{n-1} \, dx_1$$

As, $x_1$ of the sample may be anywhere between 0 to $\omega$ the mean of the minimum values of samples reduces to

$$\bar{x}_1 = \int_0^\omega p_2 \cdot x_1$$

$$= \int_0^\omega \frac{n}{\omega^n} \cdot (\omega - x_1)^{n-1} \, x_1 \cdot dx_1, \text{ using } x_1 = \omega y,$$

$$= \int_0^1 n \cdot \omega \, (1-y)^{n-1} \, y \, dy,$$

$$\therefore \quad \bar{x}_1 = \frac{1}{(n+1)} \cdot \omega.$$

which is same as the result (4)

Also $\nu_m$, the $m$th moment becomes,

$$\nu_m = \int_0^\omega p_2 \cdot (x_1 - \bar{x}_1)^m$$

$$= \int_0^\omega \frac{n}{\omega^n} \cdot (\omega - x_1)^{n-1} \left(x_1 - \frac{1}{n+1} \cdot \omega\right)^m dx_1$$

Substituting $x_1 = \omega y$,

$$\nu_m = n \cdot \omega^m \int_0^1 (1-y)^{n-1} \left(y - \frac{1}{n+1}\right)^m dy \qquad \ldots \quad (8)$$

This equation is the same as the one obtained in (5). So by integrating this, we obtain the value of $\nu_m$ to be the hypergeometric series, derived in the equation (6).

Since the distribution of the extreme values are symmetrical, the expressions $\mu_m$ and $\nu_m$ of the $m$th moments derived are identical except with the difference of $(-1)^m$. This gives alternately positive and negative moments (the odd moments being negative), for the maximum values, and all the moments positive for the minimum values.

The term $(-1)^m$ is explained by the fact that the frequency distribution for the maximum values starts at the origin and has the mode at $\omega$ and the frequency distribution for the minimum values has the mode at the origin and ends at $\omega$. So, the frequency curves indicating both the distributions are the reflections of each

other in the interval 0 to $\omega$. We further note that the constants $\beta_1$ and $\beta_2$ for both distributions are the same being,

$$\beta_1 = \frac{4\,(n-1)^2\,(n+2)}{n\,(n+3)^2} \qquad \dots \quad (9)$$

$$\beta_2 = \frac{3\,(n+2)(3n^2-n+2)}{n\,(n+3)(n+4)} \qquad \dots \quad (10)$$

We will now study the changes in the values of the constants for examples of different sizes. The following table gives the values of arithmetic mean variance, $\beta_1$ and $\beta_2$ for samples of sizes, 2, 5, 10 15 and 20, the unit in each case being $\omega$, the range of the original rectangular population.

*Constants of the distribution of extreme values.*

| Size of samples | Arithmetic Mean | | Variance | $\beta_1$ | $\beta_2$ |
|---|---|---|---|---|---|
| | Maximum | Minimum | | | |
| 2 | 0·6 | 0·3 | 0·05 | 0·320 | 2·400 |
| 5 | 0·83 | 0·16 | 0·0198 | 1·400 | 4·200 |
| 10 | 0·90 | 0·09 | 0·0069 | 2·301 | 5·776 |
| 15 | 0·938 | 0·063 | 0·0034 | 2·742 | 6·464 |
| 20 | 0·952 | 0·048 | 0·0021 | 3·026 | 7·061 |

We observe from the table that as the size of the sample increases, the A.M. increases for the distribution of the maximum values, and decreases for minimum values and the variance also decreases. The values of $\beta_1$ and $\beta_2$ increase and always fall in the Type IX region of the diagram of Karl Pearson types of curves. Here is an example where with the increase in size of a sample the distribution of the statistic deviates more from the Normal curve rather than approach it.

*Experimental Verification:*

500 samples of size 10 each are taken from the table of normal deviates published by Prof. P. C. Mahalanobis.[3]. For the maximum and minimum values the appropriate permille values are found from the tables for Statisticians and Biometricians, Part I. These latter values give the extreme values for the samples which follow

the rectangular law. These 500, maximum and minimum values are tabulated in the form of frequency distributions of 16 classes, the length of class interval in both cases being 0·03.

The following tables give the actual frequency distribution together with the theoretical frequency computed from the equations[2]

$$y = 150\ (1 + x)^9, \text{ for maximum values}$$

$$y = 150\ (1 - x)^9, \text{ for minimum values.}$$

obtained from the general equations already given.

TABLE I.

MAXIMUM VALUES

| Class Interval | Expected Frequency E | Observed Frequency O | | $\frac{(O-E)^2}{E}$ |
|---|---|---|---|---|
| 0·520—0·550 | | 1 | | |
| 0·550—0·580 | | — | | |
| 0·580—0·610 | | 1 | | |
| 0·610—0·640 | 13·99 | 2 | 16 | 0·29 |
| 0·640—0·670 | | 2 | | |
| 0·670—0·700 | | 10 | | |
| 0·700—0·730 | 7·35 | 6 | | 0·25 |
| 0·730—0·760 | 10·66 | 11 | | 0·01 |
| 0·760—0·790 | 15·17 | 9 | | 2·51 |
| 0·790—0·820 | 21·38 | 24 | | 0·32 |
| 0·820—0·850 | 29·71 | 31 | | 0·06 |
| 0·850—0·880 | 40·81 | 49 | | 1·64 |
| 0·880—0·910 | 55·46 | 48 | | 1·00 |
| 0·910—0·940 | 74·50 | 78 | | 0·16 |
| 0·940—0·970 | 99·41 | 98 | | 0·02 |
| 0·970—1·000 | 131·29 | 130 | | 0·01 |
| | | 500 | | 6·27 |

$\chi^2 = 6·27.$ $P = 0·79101.$

TABLE II.

MINIMUM VALUES

| Class Interval | Expected Frequency E | Observed Frequency O | | $\frac{(O-E)^2}{E}$ |
|---|---|---|---|---|
| 0·000—0·030 | 131·29 | 139 | | 0·45 |
| 0·030—0·060 | 99·41 | 105 | | 0·31 |
| 0·060—0·090 | 74·50 | 68 | | 0·57 |
| 0·090—0·120 | 55·46 | 53 | | 0·11 |
| 0·120—0·150 | 40·81 | 35 | | 0·83 |
| 0·150—0·180 | 29·71 | 22 | | 2·00 |
| 0·180—0·210 | 21·38 | 26 | | 1·00 |
| 0·210—0·240 | 15·17 | 14 | | 0·09 |
| 0·240—0·270 | 10·66 | 12 | | 0·17 |
| 0·270—0·300 | 7·35 | 8 | | 0·06 |
| 0·300—0·330 | | 9 | | |
| 0·330—0·360 | | 3 | | |
| 0·360—0·390 | 13·99 | 4 | 18 | 1·15 |
| 0·390—0·420 | | 1 | | |
| 0·420—0·450 | | 1 | | |
| 0·450—0·480 | | — | | |
| | | 500 | | 6·74 |

$\chi^2 = 6·74$. $P = 0·74880$.

Also the observed and the theoretical values of arithmetic mean, $\mu_2, \mu_3, \mu_4$ and $\beta_1, \beta_2$ are tabulated in the following table of 'Frequency Constants' for the extreme values of samples of 10 from a rectangular population. (unit $\omega/10$).

| 'Constant' | Maximum Values. Expected. | Maximum Values. Observed. | Minimum Values. Expected. | Minimum Values. Observed. |
|---|---|---|---|---|
| Arithmetic Mean | 9·091 | 9·087 | 0·909 | 0·912 |
| $\mu_2$ | 0·689 | 0·645 | 0·689 | 0·711 |
| $\mu_3$ | - 0·867 | - 0·721 | 0·867 | 0·836 |
| $\mu_4$ | 2·740 | 2·066 | 2·740 | 2·394 |
| $\beta_1$ | 2·301 | 1·939 | 2·301 | 1·943 |
| $\beta_2$ | 5·776 | 4·965 | 5·776 | 4·735 |

We note from the above tables that the observed frequencies are in close correspondence with the expected frequencies of extreme values and also the frequency constants closely agree with their theoretical values. Hence it is evident that the frequency distributions of the extreme values of small samples taken from a rectangular population are of the type IX curve of Karl Pearson.

So the $m$th moments $\mu_m$ and $\nu_m$ of the maximum and minimum values of samples of size $n$ taken from an infinite rectangular population of range $\omega$, about their respective arithmetic means, are given by the hypergeometric functions,

$$\mu_m = \left(\frac{\omega}{n+1}\right)^m \cdot F(-m, 1; n+1; n+1)$$

$$\nu_m = (-1)^m \cdot \left(\frac{\omega}{n+1}\right)^m \cdot F(-m, 1; n+1; n+1)$$

respectively, and both the distributions follow the type IX, curve of Karl Pearson.

## References

(1) PROF. KARL PEARSON—'On the mean character and variance of a ranked individual, and on the mean and variance of the intervals between ranked individuals':

*Biometrika*, Vol. 23, page 364 to 397.

(2) MR. N. SUNDARARAMA SASTRY, 'The Range of Samples taken from a rectangular population':

*Journal of the Indian Mathematical Society*, New series, Vol. 1, No. 7, page, 228-234.

(3) PROF. P. C. MAHALANOBIS etc., 'Tables of Random samples from a normal population':

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# ON DEMLO NUMBERS

BY

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1. A Demlo number is a number whose digits (in order) can be divided into three parts such that the sum of the first and the third parts consists of a single repeated digit, while the middle part is either empty or is a number consisting of the same repeated digit. The digit which is repeated in the above definition may be called the *constituent* of the Demlo number. Examples :

(i) 2354 is a Demlo number with 7 as the constituent digit and middle part empty, since $23+54=77$.

(ii) 9855457 is a Demlo number with 5 as the constituent digit, since $98+457=555$.

1·1. The object of this note is to prove some results concerning products of two numbers leading to Demlo numbers. We shall denote by $(p)^m$, $p \leqslant 9$, a number consisting of $p$ repeated $m$ times. We shall denote by $\left\{\frac{N}{9}\right\}$ the integral part $\left[\frac{N}{9}\right]$ of $\frac{N}{9}$ when N is not divisible by 9 and $\frac{N}{9}-1$ when N is divisible by 9. Hence we can always write

$$\frac{N}{9}=\left\{\frac{N}{9}\right\}+\frac{a}{9}, \qquad \ldots (1)$$

where $1 \leqslant a \leqslant 9$.

2. We shall prove the following

THEOREM 1. If N is a number consisting of $k$ digits and $P=(1)^l$ where $l \geqslant k$, then $N \times P$ is a Demlo number with the middle part consisting of $a$ repeated $(l-k)$ times where $a$ is defined by (1). More precisely,

$$N \times P=\left\{\frac{N}{9}\right\}(a)^{l-k}\left[(a)^k-\left\{\frac{N}{9}\right\}\right]. \qquad \ldots (2)$$

2·1. To prove this theorem we require the following.

LEMMA 1. If N is a number of $k$ digits and $a$ is defined by (1), then $\left\{\frac{N}{9}\right\}<(a)^k$.

*Proof.* Let $m = \left\{\frac{N}{9}\right\}$. If $m \geqslant (a)^k$, then (1) gives

$$N = 9m + a \geqslant 9\,(a)^k + a$$
$$= a\,\{(9)^k + 1\}$$

so that, since $a \geqslant 1$, N consists of more than $k$ digits which is contrary to hypothesis.

**2·2.** *Proof of Theorem* 1. Let $m = \left\{\frac{N}{9}\right\}$. We have

$$\begin{aligned} N \times P &= (9 \times m + a)\,(1)^l \\ &= m \times (9)^l + (a)^l \\ &= m(0)^l - m + (a)^l \\ &= m(0)^l + (a)^{l-k}\,(0)^k + \{(a)^k - m\} \\ &= m(a)^{l-k}\,\{(a)^k - m\} \end{aligned}$$

which is the result (2). Here we use the result of lemma 1 that $m < (a)^k$ so that $(a)^k - m$ is a positive integer with not more than $k$ digits. It may also be noted that *the first and the last part in the Demlo number* (2) *depends only on N and k and is independent of l.*

**2·3.** *Examples :*

(i) N = 887. Here $a = 5$ and $\left\{\frac{N}{9}\right\} = 98$, so that

$887 \times (1)^m = 98\,(5)^{m-3}\,457$ for all $m \geqslant 3$.

(ii) N = 943. Here $a = 7$ and $\left\{\frac{N}{9}\right\} = 104$, so that

$943 \times (1)^m = 104\,(7)^{m-3}\,673$ for all $m \geqslant 3$.

(iii) N = 729. Here $a = 9$, $\left\{\frac{N}{9}\right\} = 80$, so that

$729 \times (1)^m = 80\,(9)^{m-3}\,919$ for all $m \geqslant 3$.

**3.** We shall give some general applications. Firstly, we shall find the product $\{(1)^m\}^2$. Let

$$\frac{(1)^m}{9} = p_m + \frac{q_m}{9}$$

where $p_m = \left\{\frac{(1)^m}{9}\right\}$ and $q_m$ corresponds to $a$ in (1). Then, by theorem 1,

$$\{(1)^m\}^2 = p_m\ \{(q_m)^m - p_m\} \qquad \ldots \quad (3)$$

which for any given $m$ reduces the product to a division by 9 and subtraction. As an example, let us take $m = 3$. Here $p_m = 12$, $q_m = 3$ so that $(111)^2 = 12321$.

Next, let us take $m = 13$. Here $p_m = 123456790123$ and $q_m = 4$. Hence we get

$$\{(1)^{13}\}^2 = 1234567901234320987654321$$

**3·1.** Next we shall find $N^2 \times (1)^l \times (1)^m$ where N is a multiple of 3. Let $N = 3t$, so that $N^2 = 9t^2$. Then

$$
\begin{aligned}
N^2 \times (1)^l \times (1)^m &= (3N \times (1)^l + t) \times t \times (1)^m - t \times {}^2(1)^m \\
&= (9 \times (1)^l + 1) \times t^2 \times (1)^m - t^2 \times (1)^m \\
&= 1\,(0)^l \times t^2 \times (1)^m - t^2(1)^m, \qquad \ldots \quad (4)
\end{aligned}
$$

Now by theorem 1, if $m \geqslant n$, the number of digits in $t^2$, then

$$t^2 \times (1)^m = \left\{\frac{t^2}{9}\right\} (a)^{m-n} \left[(a)^n - \left\{\frac{t^2}{9}\right\}\right] = \text{R, say}, \qquad \ldots \quad (5)$$

where

$$\frac{t^2}{9} = \left\{\frac{t^2}{9}\right\} + \frac{a}{9}$$

as in (1): Let $s$ be the number of digits in R and let $l \leqslant s$. Then, by (4), we get

$$N^2 \times (1)^l \times (1)^m = (R-1)(9)^{l-s}\, R', \qquad \ldots \quad (6)$$

where $R - 1 + R' = (9)^s$. The relation (6) gives the product.

Examples: (i) Let $N = 39$, $t = 13$, $t^2 = 169$. If $m = 4$, we get

$$R = 18\ (7)^{4-3}\ (759) = 187759$$

which gives $s = 6$. Hence if $l \geqslant 6$, say $l = 10$ we get

$$39^2 \times (1)^{10}\ (1)^4 = 1877589999812241.$$

(ii) Let $N = 93$, $t = 31$, $t^2 = 961$. Here $n = 3$. Take $m = 5$. Then

$$R = 10677671,$$

so that $s = 8$. Take $l = 12$. Then we get

$$9 \times 961 \times (1)^{12} \times (1)^5 = 1067767099999893223329$$

The above results contain as particular cases some of the empirical results obtained by D. R. Kaprekar.* It may be also noted that the definition of a Demlo number adopted here is slightly more general than that adopted by Mr. Kaprekar.

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## GLEANINGS

Through all this ordeal his root horror had been isolation, and there are no words to express the abyss between isolation and having one ally. It may be conceded to the mathematicians that four is twice two. But two is not twice one; two is two thousand times one. That is why, in spite of a hundred disadvantages, the world will always return to monogamy.

G. K. CHESTERTON in *The Man who was Thursday.*

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* D. R. Kaprekar, "Products of some linear Demlo numbers". *The Jour. University of Bombay.* Vol. VIII, part 3, pp. 1-12.

# A STATISTICAL ANALYSIS OF THE CORRESPONDENCE BETWEEN THE M. A.-M. Sc.. AND B. A.-B. Sc. EXAMINATION RESULTS OF THE CALCUTTA UNIVERSITY IN PURE MATHEMATICS

BY

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## Introduction

One may be interested to know how the arts and science graduates in Mathematics are expected to fare in the examination for the Master's degree in Pure Mathematics. What percentage of marks will be obtained in the M. A.–M. Sc. examination by a candidate, getting a certain percentage of marks in the B. A.–B. Sc. examination; whether a second or a third attempt will show any improvement in marks; if some knowledge of sister-sciences will be of any help for the mastery of Pure Mathematics—these are interesting studies. All these considerations will also help the young generation to decide whether to take up Pure Mathematics or choose some other line. Thinking that a statistical investigation into the questions, mentioned above, may be important for the University authorities and the educationists of our country, I undertook this investigation under the direction of Professor F. W. Levi, Hardinge Professor of the Calcutta University. One of the facts, revealed by the present investigation, is that during the last three years none of the 26 candidates, who got less than 60% in Mathematics in pass papers in the bachelor's degree examination, succeeded in passing the M. A.–M. Sc. examination in Pure Mathematics. Even allowing for errors of sampling we can safely infer (5% level) from this that in the long run not more than 11% of candidates, getting less than 60% in the B. A.–B. Sc. pass papers, will pass the M. A.–M. Sc. examination, even if they make a second or a third attempt. This shows how very necessary it is to impose some restriction on the admission of students into the post-graduate classes in Pure Mathematics.

## 1. The data

As the new syllabus for the M. A.–M. Sc. examination in Mathematics came in operation in 1938, the analysis has been confined to

only those candidates who appeared for the Master's degree in Pure Mathematics during the years 1938–1940. The total number of candidates during these three years was 123 (counting every candidate as one, even though he might have appeared more than once during these three years). Information was collected under the following heads for each of these candidates, except 4 who had graduated from other Indian Universities, and about whom the information under (iv) was not available.

(i) Register number with year.

(ii) Whether Arts or Science student (i.e. whether appeared for the M. A. or for the M. Sc. examination).

(iii) Year or years of appearance for the M. A. or M. Sc. examination with corresponding marks (it should be remembered that those who first appeared in 1938, had the chance of reappearing during 1939 or 1940; those who first appeared in 1939 had the chance of reappearing again in 1940; and those who first appeared in 1940 had no chance of reappearing again).

(iv) Year of appearance for the B. A. or. B. Sc. examination (pass or honours) with corresponding marks in Mathematics only.

Our analysis is thus confined to 119 candidates, distributed as follows :—

TABLE I

| | (i) First appearance. | (ii) Appeared in the yrs. following first appearance. | (iii) Appeared 2 yrs. after first appearance. | (iv) Total |
|---|---|---|---|---|
| 1938 | 20 (0) P<br>9 (7) H | | | 20 (0) P<br>9 (7) H |
| 1939 | 36 (19) P<br>15 (9) H | 13 (0) P<br>3 (3) H | | 49 (19) P<br>18 (12) H |
| 1940 | 28 (7) P<br>11 (7) H | 21 (11) P<br>0 (0) H | 4 (0) P<br>0 (0) H | 53 (18) P<br>11 (7) H |
| Total | 84 (26) P<br>35 (23) H | 34 (11) P<br>3 (3) H | 4 (0) P<br>0 (0) H | |

Explanation :—The figures within brackets indicate arts, P for pass and H for honours. For example, looking at the figures for 1940 under column (iv), we find that 53 pass and 11 honours* graduates appeared this year, of whom 18 and 7 respectively were arts students (i.e. came from B. A.). Similarly looking at the figures for 1939 under column (ii), we find that among the candidates who first appeared in 1938, 16 reappeared in 1939, of which 13 were pass students (all science graduates) and 3 honours students (all arts graduates).

When comparing the performance of the candidates at the B. A.–B. Sc. and M. A.–M. Sc. examinations it must be borne in mind that due to the smallness of the number of candidates, fine grouping would be out of place, therefore, as regards their performance in the B. A.–B. Sc. examination, the candidates have been grouped throughout under the following heads:

(i) below 60% in pass,

(ii) 60% and above but below 70% in pass,

(iii) 70% and above in pass,

(iv) below 40% in honours (i. e. sat for honours without getting it),

(v) 40% and above but below 60% in honours (i.e. 2nd class honours),

(vi) 60% and above in honours (i. e. 1st class honours);

and as regards their performance in the M. A.–M. Sc. examination, under the following heads:

(i) below 36% (i. e. failed),

(ii) 36% and above but below 45% (i. e. 3rd class),

(iii) 45% and above but below 60% (i. e. 2nd class),

(iv) 60% and above (i. e. 1st class).

## 2. Comparison between the performance of arts and science students.

A superficial examination of the data shows that the arts students have done much better in the examination for the Master's degree than the science students, as the following table (containing figures for the first attempt only) would show.

* By honours student we shall always mean, those who sat for honours whether he gets it or not.

TABLE II—A.

Performance in the M. A.—M. Sc. Examination.

| | Failed | 3rd class | 2nd class | 1st class | Total. |
|---|---|---|---|---|---|
| Arts. | 29<br>=59·2% | 5<br>=10·2% | 8<br>=16·3% | 7<br>=14·3% | 49<br>=100% |
| Science. | 56<br>=80% | 5<br>=7·1% | 5<br>=7·1% | 4<br>=5·8% | 70<br>=100% |

In order to examine the matter more closely, we should however see how the 119 candidates whose performance (at the first attempt) in the M. A.–M. Sc. examination has been shown above were recruited (i. e. what was their performance at the B. A.–B. Sc. examination). This is given in the following table:

TABLE II—B.

Performance in the B. A.—B. Sc. Examination.

| | Below 60% in pass. | 60% and above but below 70% in pass. | 70% and above in pass and below 40% in Hons. | 2nd class Honours. | 1st class Honours. | Total. |
|---|---|---|---|---|---|---|
| Arts. | 5<br>=10·2% | 13<br>=26·6% | 8+6<br>=28·5% | 14<br>=28·6% | 3<br>=6·1% | 49<br>=100% |
| Science. | 21<br>=30% | 18<br>=25·7% | 19+2<br>=30% | 8<br>=11·4% | 2<br>=2·9% | 70<br>=100% |

This clearly shows that a much better type of Arts students join the post-graduate classes in Pure Mathematics, than the Science students.* The following statistical investigation shows that the better performance of the Arts students is completely accounted for by this cause. In fact there can be detected no difference between the performance of Arts and Science students in the examination for the Masters degree, when we take into account their performance in examination for the Bachelor's degree. We first form the following tables:

* Unlike other science subjects, there is no "numerus clausus" in Pure Mathematics; sometimes those candidates who are refused seats in post-graduate classes in other science subjects, get themselves admitted in Pure Mathematics.

TABLE II—C.

ARTS STUDENTS

Performance in M. A. (First attempt).

| Performance in B. A. | Failed | 3rd class | II class | I class | Total |
|---|---|---|---|---|---|
| Below 60% in pass. | 5 | 0 | 0 | 0 | 5 |
| 60% and above but below 70% in pass. | 11 | 1 | 1 | 0 | 13 |
| 70% and above in pass. | 6 | 1 | 1 | 0 | 8 |
| Below 40% in Hons. | 5 | 1 | 0 | 0 | 6 |
| II class Hons. | 2 | 2 | 6 | 4 | 14 |
| I class Hons. | 0 | 0 | 0 | 3 | 3 |

TABLE II—D.

SCIENCE STUDENTS.

Performance in M. Sc.—(First attempt).

| Performance in B. Sc. | Failed | III Class | II Class | I Class | Total. |
|---|---|---|---|---|---|
| Below 60% in pass. | 21 | 0 | 0 | 0 | 21 |
| 60% and above but below 70% in pass. | 17 | ...1 | 0 | 0 | 18 |
| 70% & above in pass. | 15 | 2 | 2 | 0 | 19 |
| Below 40% in Hons. | 2 | 0 | 0 | 0 | 2 |
| II Class Hons. | 1 | 2 | 3 | 2 | 8 |
| I Class Hons. | 0 | 0 | 0 | 2 | 2 |

A glance at the above table shows that candidates with a similar performance at the B. A.–B. Sc. examination give a similar performance at the M. A.–M. Sc. examination, irrespective of whether they are Arts or Science candidates. Thus all the 5+21 candidates getting below 60% in pass B. A. or B. Sc. failed in the M. A. or M. Sc. Similarly all the 3+2 candidates getting a first class honours in B. A. or B. Sc. got a first class in the M. A. or M. Sc. For these two groups of candidates the identity is complete. This complete identity cannot, of course, be expected for every group, due to the intervention of random fluctuations, but the correspondence is still quite close, and we can use the $\chi^2$-statistics to test whether there is any significant difference in the structure of the tables II C and II D. In order that the frequency in the cells be not too small, we amalgamate the third class with the second class for M. A. and M. Sc. and the pass students getting above 70% with honours students getting below 40% for the B. A. and B. Sc. (the amalgamated cells are bounded by thick lines in the table II C and II D). Expected values in every cell are now calculated, on the hypothesis that there is no difference in the structure of the two tables. For example there were 13+18=31 candidates who got above 60% but below 70% in pass B. A. or B. Sc. and out of them 11+17=28 failed. We should have expected $(28/31)\times 13 = 11{\cdot}7$ of these candidates to be Arts students and $(28/31)\times 18 = 16.3$ to be Science students. Now calculating $\chi^2$ by the well-known formula

$$\chi^2 = \sum \frac{(\text{observed-expected})^2}{\text{observed}},$$

it is found that $\chi^2 = 1.12$. As there are four degrees of freedom, the probability of this value of $\chi^2$ is greater than ·8. Thus the observed departure from expectation is so small that it would arise by chance in 80% cases. Hence we can safely conclude that there is no difference in the performance of Arts and Science students in the examination for the Master's degree when their performance in the examination for the Bachelor's degree is taken into account.

### 3. Comparison between different years

We can in a similar manner settle the question, whether there has been any significant change in the standard of the M. A.–M. Sc. examination during the three years under study, relative to the standard of the B. A.–B. Sc. examination. It can be shown that the answer must be negative. To do this we form tables III A, III B, III C, showing the performance in the B. A.–B. Sc. and M. A.–M. Sc. of the students who appeared for the later examination (for the first time) in 1938, 1939 and 1940 respectively.

TABLE III—A.
1938
Performance in the M. A.—M. Sc. Exam.

Performance in B. A.—B. Sc. Exam.

| | Failed | III | II | I | Total |
|---|---|---|---|---|---|
| Below 60% in pass | 9 | 0 | 0 | 0 | 9 |
| 60%—70% | 5 | 1 | 0 | 0 | 6 |
| Above 70% in pass | 5 | 0 | 0 | 0 | 5 |
| Below 40% in Hons. | 2 | 0 | 0 | 0 | 2 |
| II Class Hons. | 1 | 0 | 3 | 2 | 6 |
| I Class Hons. | 0 | 0 | 0 | 1 | 1 |

TABLE III—B.
1939
Performance in M. A.—M. Sc. Examination.

Performance in B. A.—B. Sc. Examination.

| | Failed | III | II | I | Total |
|---|---|---|---|---|---|
| Below 60% in pass | 6 | 0 | 0 | 0 | 6 |
| 60%—70% in pass | 13 | 0 | 1 | 0 | 14 |
| Above 70% in pass | 13 | 2 | 1 | 0 | 16 |
| Below 40% in Hons. | 4 | 1 | 0 | 0 | 5 |
| 2nd Class Hons. | 1 | 2 | 5 | 1 | 9 |
| I Class Hons. | 0 | 0 | 0 | 1 | 1 |

TABLE III—C.
1940
Performance in M. A.—M. Sc. Examination.

| | | Failed | III | II | I | Total |
|---|---|---|---|---|---|---|
| Performance in B. A.—B. Sc. Examination. | Below 60% in pass | 11 | 0 | 0 | 0 | 11 |
| | 60%—70% in pass | 10 | 1 | 0 | 0 | 11 |
| | Above 70% in pass | 3 | 1 | 2 | 0 | 6 |
| | Below 40% in Hons. | 1 | 0 | 0 | 0 | 1 |
| | II Class Hons. | 1 | 2 | 1 | 3 | 7 |
| | I Class Hons. | 0 | 0 | 0 | 3 | 3 |

We can calculate expected values on the hypothesis that there is no significant difference between the three years after amalgamating the cells as in §2. Calculating $\chi^2$ as before we find that $\chi^2 = 7{\cdot}17$. Since there are now 8 degrees of freedom, the probability of this value of $\chi^2$ is greater than ·7. Thus the observed departure from expectation would arise by chance in 50% cases, and we may conclude that it is not significant.

**4. Association between the performance in the B.A.-B.Sc. and M. A.—M. Sc. examinations (first attempt).**

We have seen that there is no significant difference between Art and Science or between the years 1938, 1939 and 1940. We are therefore justified in pooling our results (i.e. considering the performance of a candidate in the M. A.—M. Sc. in relation to his performance in B. A.—B. Sc., regardless of whether he was an Arts or Science student, or of the year in which he appeared).

We have now to consider whether a better performance in the B. A.—B. Sc. is associated with a better performance in the M. A.—M. Sc. Consider first the case of pass students only (as regards their first attempt). The results are shown in the following table.

TABLE VI A.
There was no first class
Pooled results for Pass Students
Performance in M. A.—M. Sc.

| Performance in B. A.—B Sc. (pass) | Failed | III | II | Total |
|---|---|---|---|---|
| Below 60% | 26 | 0 | 0 | 26 |
| 60%—70% | 28 | 2 | 1 | 31 |
| 70% and above | 21 | 3 | 3 | 27 |
| Total | 75 | 5 | 4 | 84 |

Calculating $\chi^2$ on the null-hypothesis we find $\chi^2 = 7{\cdot}19$ so that with 4 degrees of freedom $P(\chi^2) = {\cdot}136$, which means that even if there is no association between the B. A.—B. Sc. and M. A.—M. Sc. marks such a result, as that observed, would arise by chance in roughly 14 cases out of 100. The association is not significant on the 10% level but can be regarded as significant on the 14% level. The evidence for association is thus inconclusive.

An entirely different state of affairs is revealed in the case of Honours students. Their results are shown in the following table :

TABLE IV—B.
Pooled results for Honours Students.
Performance in the M. A.—M. Sc.

| Performance in the B. A.—B. Sc. Hons. | Failed | III | II | I | Total. |
|---|---|---|---|---|---|
| Below 40% | 7 | 1 | 0 | 0 | 8 |
| II Class | 3 | 4 | 9 | 6 | 22 |
| I Class | 0 | 0 | 0 | 5 | 5 |
| Total | 10 | 5 | 9 | 11 | 5 |

Calculating $X^2$ on the null-hypothesis we find that $X^2 = 29{\cdot}05$ so that with 6 degrees of freedom $P(X^2) = {\cdot}0001$. Thus such a distribution, as observed, will arise by chance, in less than one case out of 10,000, if there is no association between the B. A.—B. Sc. and M. A.—M. Sc. marks. We, therefore, find that in the case of honours students there is a highly significant association between the performance in B. A.—B. Sc. and the M. A.—M. Sc. One of the reasons for this striking divergence between pass and honours seems to be that while the position of the honours student in Mathematics in the B. A.—B. Sc. examination is solely determined by his result in Mathematics, that of the pass student is determined by his results in 3 or 4 subjects; so that the honours student concentrates mainly on Mathematics, the pass student has to distribute his efforts equally over three or four subjects of which Mathematics is only one. Thus the intrinsic difference in mathematical ability between two pass students does not show up so much in the marks, obtained by them, as would have been the case, if they would concentrate on Mathematics. So the result in Mathematics of pass students is not a satisfactory guide for judging their performance in the M.A.—M.Sc. examination.

If we know the B. A.—B. Sc. result of a candidate, what can we predict about his M. A.—M. Sc. result? In the case of pass students, or those who sat for honours without securing it we can not do much more than to calculate the chance of passing. In table IV-C

TABLE IV—C.

| Performance in B. A.-B. Sc. | (1) Total No. of candidates | (2) Percentage passed in M. A.-M. Sc. | (3) Confidence interval at 5% level. |
|---|---|---|---|
| Below 60% in pass | 26 | 0% | ——11% |
| 60%-70% in pass | 31 | 9·7% | 1%——23% |
| 70% & above in pass | 27 | 22·2% | 10%——39% |
| Below 40% in Hons. | 8 | 12·5% | ·7%—— 47% |

in the column (2) is given the actual percentage of passes in the M. A.—M. Sc. examination, of candidates falling in a given category with respect to their result in the B. A.—B. Sc. examination. For example, among the candidates who got above 70% in the pass B. A.—B. Sc. examination, 22·2% actually passed in the M. A.—M. Sc. examination. The percentages of the column (2) give also the best estimate (on the basis of available materials) to the percentage of passes in the M. A.—M. Sc. examination for a candidate of a given category (with respect to his B. A.—B. Sc. result). Thus supposing that the examination was continued without change of standards, the best estimate we could give of the average percentage of passes in the M. A.—M. Sc. from among the candidates who would get about 70% in pass B. Sc. would be 22·2%. Due to presence of sampling errors this estimate however differs from the true percentage of which it is an estimate. Column (3) of the table gives the confidence interval at 5% level. Thus we can assert, at the risk of being wrong once in twenty times, that the true percentage of passes in the M. A.—M. Sc. examination from among the candidates who get above 70% in the B. A.—B. Sc. examination would lie between 10% and 39%.

In the case of candidates who actually obtained an Honours degree in B. A.—B. Sc., we can give a slightly more precise result, and try to estimate the actual marks in the M. A.—M. Sc. on the basis of their marks in the B. A.—B. Sc. examination, since there is a very clear evidence that increasing marks in the B. A.—B. Sc. are associated with increasing marks in the M. A.—M. Sc. In the graph (fig. 1), points show the marks, obtained by the individual candidates at the two examinations. The line of regression

$$y = 1{\cdot}63\ x - 60{\cdot}3,$$

giving the regression of the M. A.—M. Sc. marks $y$ on the B. A.—B. Sc. marks $x$, is also shown. The usual normal theory does not apply, since the data is not at all normal. But the marks of 22 out of 27 candidates lie within the limits $y \pm 100$, i.e.

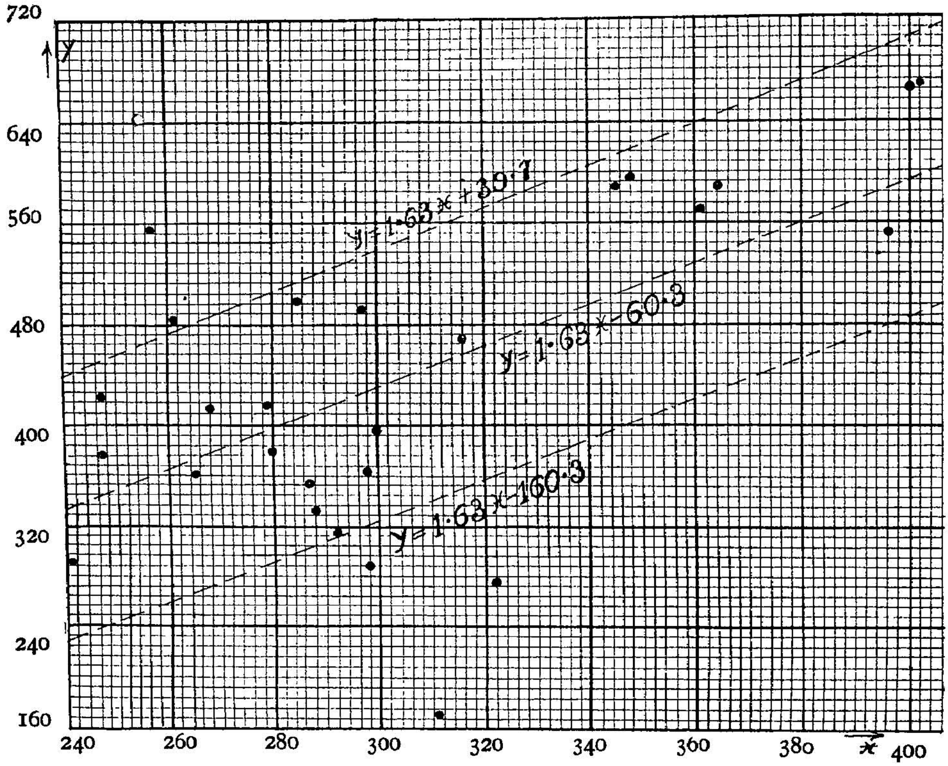
$$1{\cdot}63x - 160{\cdot}3 \text{ to } 1{\cdot}63x + 39{\cdot}7.$$

We may, therefore, estimate that in general in 81·5% cases, the M. A.—M. Sc. marks of candidates, getting $x$ marks in the B. A.—B. Sc. examination would lie between the above limits. The 5% confidence interval for this percentage is 65%—92%.

## 5. Second and third attempt

Among the candidates who failed in the first attempt in the M. A.-M. Sc. examination, a certain number made a second attempt. Their actual results are summarised in the following table:—Further

FIGURE 1




statistical analysis would not yield any useful results on account of the small numbers involved.

TABLE V

| | Total number of candidates in '38 and '39 | Passed in the 1st attempt in M.A.–M.Sc. with percentage | No. making 2nd attempt out of a possible total of | Passed in the 2nd attempt in M.A.–M.Sc. | Percentage of passes in the first and 2nd attempt |
|---|---|---|---|---|---|
| Below 60% (pass) | 15 | 0=0% | 11 out of 14 | 0 | 0% |
| 60%—70% (pass) | 20 | 2=10% | 14 out of 18 | 1 | 15% |
| Above 70% (pass) | 21 | 3=14·3% | 10 out of 18 | 3 | 28·6% |
| Below 40% (Hons.) | 7 | 1=14·3% | 2 out of 6 | 0 | 14·3% |
| 40%—60% (Hons.) | 15 | 13=86·7% | 1 out of 2 | 1 | 93·3% |
| Above 60% (Hons.) | 2 | 2=100% | 0 out of 0 | 0 | 100% |
| Total | 80 | 21 | 38 out of 58 | 5 | 32·5% |

A third attempt was possible only for those who appeared first in 1938, and failed both in 1938 and 1939. There were 13 such candidates for which only three made the third attempt and all failed.

## Summary

The results of 119 candidates who appeared for the M. A.–M. Sc. examination in Pure Mathematics during the three years 1938 to 1940, were analysed in relation to their performance (in Mathematics) in the B. A.–B. Sc. examination : The following conclusions were arrived at:—

(1) The arts students who appeared for the M. A. examination during the years 1938–40 were of distinctly better type, as judged by their performance both in the M. A. and B. A. than the corresponding science students who appeared for the M. Sc. examination; but there is no intrinsic difference between arts and science students in as much as arts and science students, belonging to the same category (judged by their performance in the B. A.–B. Sc.) achieved similar results in the M. A.–M. Sc.

(2) There was no distinction between the years 1938, 1939 and 1940, so that the standard of M. A.–M. Sc. examination during these three years can be considered to be fairly steady.

(3) Of the 84 M. A.–M. Sc. candidates who obtained the pass bachelor degree in B. A.–B. Sc., only 9 passed in the M. A.–M. Sc. in the first attempt and none among the 26 candidates who got below 60% in pass succeeded in passing. Of the 8 candidates who sat for Honours B. A.–B. Sc. but secured only a pass degree, only one passed in the M. A.–M. Sc. The percentages of passing in the M. A.–M. Sc. for students belonging to various categories with respect to their results in the B. A.-B. Sc., are shown in table IV-C together with the confidence interval at 5% level.

(4) Among the 22 candidates who got a second class honours in B. A.–B. Sc., three failed, 4 got a third class, 9 a second class, and 6 a first class in the M. A.–M. Sc., while all the five candidates who got a first class honours degree secured a first class in the M. A.–M. Sc. If $x$ represents the honours marks in the B. A.–B. Sc., the marks in the M. A.–M. Sc. of 81.5% of the students lie within the limits

$$(1{\cdot}63x - 60{\cdot}3) \pm 100,$$

the confidence interval at 5% level for the percentage being 65%–92%.

(5) Of 58 candidates for whom it was possible to make a second attempt, 38 appeared for the second time, but only 5 succeeded in passing. For more detailed information see table V. Of the 13 candidates who could make a third attempt only 3 did so, but none came out successful.

In writing this paper I have been greatly helped by the suggestions and guidance of Mr. R. C. Bose, Lecturer of our department. Here I take the opportunity of thanking him for his kind help.

CALCUTTA UNIVERSITY

*Calcutta*

# NOTES AND DISCUSSIONS

## A further note on Feuerbach's theorem

In the interesting and detailed article of Mr. A. A. Krishnaswamy Ayyangar on page 16 of *The Mathematics Student* Vol. IX (March 1941) he points out in para 4, that a really elementary proof ought not to make use of ratios or trigonometrical formulae and it should be a pure 'angle chaser.' He then gives an elementary proof provided to us by W. F. Beard (*Math. Gaz.* 1934) modified and condensed by him.

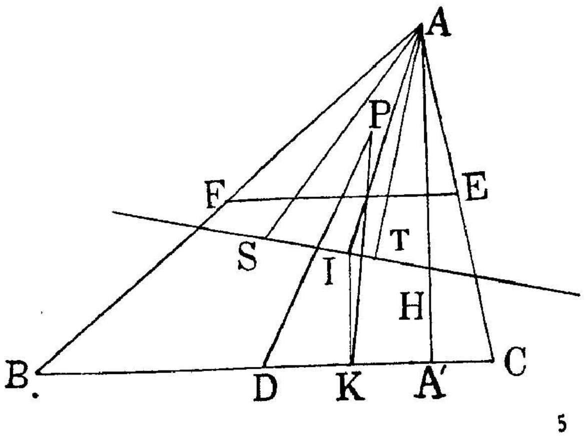
In this proof (1) the in-Feuerbach point P is obtained as the other point of intersection of the nine-point-circles of BIC, CIA (I use Mr. Krishnaswamy Ayyangar's notation in the article) one point of intersection being already known, viz, the middle point (W) of IC.

(2) It is next shown that the nine-point-circle passes through P and also the in-circle passes through P.

(3) It is then shown that $D\hat{P}A' = 2\ K\hat{P}A'$ where D is the middle point of BC, A′ is the foot of the altitude from A on BC and K is the point of contact of the in-circle with BC. It then follows that the two circles touch each other.

But there is a simpler method of obtaining the in-Feuerbach point P. Take the line SI. Draw AT perpendicular to SI and take the reflection (P) of T in EF where E and F are the middle points of AC and AB.

Now, so early as December 1904, Mr. C. E. Youngman, showed that such a point like P obtained as above has the following properties :



(1) If any point J is taken on SI and JX, JY, JZ are drawn perpendicular to the sides, then the triangles PDX, PEY, PFZ are similar to ASJ, BSJ, CSJ respectively.

(2) The circle XYZ passes through P. It follows that if we take I for J, the in-circle passes through P and if we take S for J the nine-point-circle passes through P.

Also $D\hat{P}K = S\hat{A}I = H\hat{A}I = K\hat{P}A'$.

Thus we get the identical three results mentioned in Mr. Krishnaswamy Ayyangar's article.

Though the rest of the proof is the same, the method of obtaining P the in-Feuerbach point by drawing two straight lines, AT and TP perpendicular to SI and EF respectively is far simpler than obtaining it as the intersection of the two nine-point-circles of BIC and CIA. Undoubtedly this proof is the simplest geometrical proof.

I may add that P is the orthople of SI with respect to the $\triangle ABC$ and that Youngman, though he deduces five interesting theorems, from his theorem, does not expressly state Feuerbach's theorem as one (Educational Times Reprint 2nd Series Vol. VI. page 101-2) The fifth of his deduced theorems is, that if any point J and its isogonal conjugate are collinear with S, then their common pedal circle touches the nine-point-circle (M'Cay *Trans. Royal Irish Academy* 1889). Feuerbach's theorem is a particular case of this, for I is its own isogonal conjugate and therefore lies on the diameter SI. Except in this way Youngman does not touch on Feuerbach's theorem. However, without going to the wider proposition of M'Cay we get the simplest proof of Feuerbach's theorem by angle-chasing. Except that the point P is obtained in two different ways there is practically no difference between this method and Beard's method.

Youngman's theorem is not limited to SI but to any line drawn through S.

*Madras.*

SIR V. RAMESAM.

## On Some Properties of Rectangular Hyperboloids

1. In Salmon's *Analytical Geometry of Three Dimensions*, page 233, is indicated a method of finding the condition, in tetrahedral coordinates, for a rectangular Hyperboloid (of the first kind) i.e. a conicoid whose asymptotic cone has three mutually perpendicular generators. In this note, this condition is obtained by transforming to rectangular co-ordinates and is written in a form from which certain results are deduced.

2. Let the equation of a conicoid referred to a tetrahedron $D_1 A_1 B_1 C_1$ be $(A, B, C, D, F, G, H, U, V, W)(x, y, z, t,)^2=0$ where the co-ordinates are such that $x+y+z+t=1$. Taking $D_1$ as the origin of rectangular co-ordinates, $D_1A_1$ as the X-axis, the perpendicular through $D_1$ on to the plane $D_1A_1B_1$ as the Z-axis, the equations of transformations are

$$x=\{X \sin \omega - Y \cos \omega - Z \cot \theta \sin (\omega-\alpha)\} / a \sin \omega$$
$$y=\{Y - Z \cot \theta \sin \alpha\} / b \sin \omega$$
$$z=Z / c \sin \theta$$
$$t=1-x-y-z$$

where $D_1A_1=a$, $D_1B_1=b$, $D_1C_1=c$, $\angle A_1D_1B_1=\omega$, $\theta$ is the angle which $D_1C_1$ makes with its projection on $Z=0$ and $\alpha$ the angle which this projection makes with $D_1A_1$. Writing $p=A+D-2U$, $q=B+D-2V$, $r=C+D-2W$ etc. and transforming to rectangular co-ordinates by the scheme given above, and equating to zero the sum of the co-efficients of $X^2$, $Y^2$, $Z^2$ we get the condition for a rectangular hyperboloid. The co-efficient of $p$ in this condition is equal to

$$\frac{\operatorname{cosec}^2 \omega}{a^2} + \frac{\cot^2 \theta}{a^2 \sin^2 \omega} \cdot \sin^2(\omega-\alpha) + \frac{\cot^2 \theta \sin \alpha \sin(\omega-\alpha)}{ab \sin^2 \omega} - \frac{\cot \theta \sin(\omega-\alpha)}{ac \sin \omega \sin \theta} - \frac{\cos \omega}{ab \sin^2 \omega}$$

$$= \frac{1}{18 V^2} \{\Delta_a^2 - \Delta_b^2 + \Delta_d^2 - \Delta_c^2 + 2\Delta_b \Delta_c \cos (\Delta_b \Delta_c)\} \qquad \ldots (2{\cdot}1)$$

where V is the volume of the tetrahedron, $\Delta_a$ the area of the face opposite to $A_1$ and $(\Delta_a \Delta_c)$ the angle between the faces $D_1B_1C_1$ and $D_1A_1B_1$. Writing

$$\Delta_a^2 = \Delta_a \Delta_c \cos(\Delta_a \Delta_c) + \Delta_a \Delta_b \cos(\Delta_a \Delta_b) + \Delta_a \Delta_d \cos(\Delta_a \Delta_d),$$

etc., (2·1) simplifies to $\dfrac{\Delta_a \Delta_d \cos(\Delta_a \Delta_d)}{9V^2}$.

Similarly the coefficients of $q, r, s, u, v, w$ can be written down and hence the condition turns out to be

$$p\Delta_a \Delta_d \cos(\Delta_a \Delta_d) + q\Delta_b \Delta_d \cos(\Delta_b \Delta_d) + r\Delta_c \Delta_d \cos(\Delta_c \Delta_d)$$
$$+ s\Delta_a \Delta_b \cos(\Delta_a \Delta_b) + u\Delta_b \Delta_c \cos(\Delta_b \Delta_c) + v\Delta_a \Delta_c \cos(\Delta_a \Delta_c) = 0 \quad \ldots \quad (2{\cdot}2)$$

If the tetrahedron be self-conjugate this condition becomes

$$A\Delta_a^2 + B\Delta_b^2 + C\Delta_c^2 + D\Delta_d^2 = 0.$$

Hence "*If a tetrahedron be self-congugate with regard to a rectangular hyperboloid, the rectangular hyperboloid passes through the centres of the eight spheres touching the four faces of the tetrahedron and conversely,*" from which we derive the results that:—

"*The quartic curve common to two rectangular hyperboloids passes through the centres of the eight spheres touching the faces of the common self-polar tetrahedron.*"

3. Now, the eight points $(\pm\Delta_a, \pm\Delta_b, \pm\Delta_c, \pm\Delta_d)$ lie on six different pairs of planes whose equations are $\frac{x}{\Delta_a} \pm \frac{y}{\Delta_b} = 0$ etc. Also they are the vertices of two tetrahedra desmic with the given one. Six of the planes pass through any of the eight points and of the six planes that pass through a second point, at least two are common with the six planes through the first point. Hence two points being arbitrarily chosen, it is always possible to choose 4 points out of the remaining six which are coplanar and the plane does not pass through either of the two chosen points. Now if two rectangular hyperboloids have a common generator, it must be a line through two and only two of the eight points. And as the plane through 4 properly chosen points cannot be intersected by the cubic curve common to the two rectangular hyperboloids in more than 3 points, it follows that "*If two rectangular hyperboloids have a common generator, they must have three others.*"

4. Through the quartic curve common to two conicoids only one rectangular hyperboloid (of the first kind) can be drawn. We shall next derive the conditions that this rectangular hyperboloid may be of one or two sheets.

If the two conicoids be $\sum Ax^2 = 0$, $\sum A'x^2 = 0$ where A, B, C, A′, B′, C′$>0$ and D, D′$<0$. Two cases arise according as

$$A/A' > B/B' > C/C' > D/D' \text{ or } A/A' > B/B' > D/D' > C/C'.$$

In the first case only two of the four cones of the system are real and in the latter case all the four are real. The condition for

a rectangular hyperboloid is $f(\lambda) \equiv \sum (A - \lambda A')\, \Delta_a^2 = 0$. Remembering that $\sum (A - \lambda A')\, x^2 = 0$ represents a hyperboloid of one or two sheets according as two or three of the coefficients are of like signs, it follows that "*If the centres of the eight spheres are on the same side of the four cones or on the same side of two of them, the rectangular hyperboloid is of two sheets; in other cases it is of one sheet.*"

Considering similarly the system of conicoids (with a common developable) $\sum \frac{x^2}{A - \lambda A'} = 0$ we get three rectangular hyperboloids of the system given by $F(\lambda) \equiv \sum \Delta_a^2\, (B - \lambda B')\, (C - \lambda C')\, (D - \lambda D') = 0$. In the first case when $A/A' > B/B' > C/C' > D/D'$ it can be easily seen that the roots of $F(\lambda) = 0$ are all real and determine two rectangular hyerboloids of two sheets and one of one sheet. In the second case, one root is always real and gives a rectangular hyperboloid of one sheet. The nature and the reality of the other two are left in doubt.

Science College, Patna. V. RANGACHARIAR.

## On The Self Polar Tetrahedra of Three Conicoids.

THEOREM:—Let $S_1$, $S_2$, $S_3$ be three conicoids and $A_{12}\, B_{12}\, C_{12}\, D_{12}$; $A_{23}, \ldots$; $A_{31}, \ldots$ the common self polar tetrahedra of the pairs $S_1, S_2$; $S_2, S_3$; $S_3, S_1$. Then, the twelve vertices of the three tetrahedra all lie on the same conicoid.

*Proof*:—A unique conicoid S may be made to pass through the nine points $A_{23}$, $B_{23}$, $C_{23}$, $D_{23}$, $A_{13}$, $B_{13}$, $C_{13}$, $A_{12}$, $B_{12}$. Since S passes through the seven vertices of the two tetrahedra $A_{23} \ldots$ and $A_{13} \ldots$ both of which are self polar w. r. t. $S_3$, it passes through the eighth vertex $D_{13}$. If $C_{12}$ and $D_{12}$ do not lie on S let $C_{12}\, D_{12}$ meet S in X, Y; and let X′ be the point on $C_{12}\, D_{12}$ conjugate to X w. r. t. $S_1$. Then $A_{13}\, B_{13}\, XX'$ is self polar w. r. t. $S_1$ and since S passes through seven of the vertices of the tetrahedra $A_{13}\, B_{13}\, XX'$, $A_{13}\, B_{13}\, C_{13}\, D_{13}$, it also passes through X′. Hence X′ coincides with Y. Thus $A_{12}\, B_{13}$ XY is self polar w. r. t. $S_1$ and similarly w. r. t. $S_2$. Therefore $A_{12}\, B_{12}$ XY is identical with $A_{12}\, B_{12}\, C_{12}\, D_{12}$. Hence the theorem.

K. O. AUGUSTINE.

# REVIEWS

RAYMOND CLARE ARCHIBALD:—*Outline of the History of Mathematics.*— Fifth Edition, Revised and enlarged. Price 75 cents. 1941. (The Mathematical Association of America, Inc. Oberlin, Ohio.)

The last edition of this work was reviewed in the *Student,* Vol. VII, pp. 34,35, and there is very little to add to what has been already said there. The reference to Asiatic mathematics continues to be meagre, though historically it has been the immediate inspiration of all medieval European mathematics and is the progenitor of all modern undergraduate arithmetic, algebra and mensuration.

Aryabhata, the greatest Indian mathematician before the sixth century A. D. achieved the remarkable feat of condensing all mathematics then known and some innovations also in 108 couplets, which include trigonometrical tables, a table of astronomical constants, a new numeral notation and a modern astronomical doctrine embedded in a convincing simile. For a brief perspective survey of Indian mathematics before the sixteenth century, we should refer to pp. 1-18, of Vol. I. of this journal. The only reference to Aryabhata in this History is found in n. 103 of the Literature List.

Feuerbach, the discoverer of one of the most beautiful theorems in elementary geometry till now known, does not find any place at all in the entire Outline. Such omissions may or may not be serious, but they are worth notice.

The second half of the Outline is dangerously growing out of all proportion to the first half, which is kept almost invariant with the minimum of verbal and textual alterations. It is also, perhaps, partly responsible for increasing the price by fifty percent. It is a moot proposition for the consumer, whether he should pay more for a sidedish than for the main item of food. We, however, fully sympathise with the author in the 'irksome limitations' which have compelled him to take refuge in LLN for making several desirable alterations.

On the whole, we cannot do better than commend this book to our readers in the spirit and manner of Bhaskara, eight centuries ago:

'Read, read, mathematician, this little work, to increase your intellect, and to ripen your wisdom; it is clothed in choice words, readily understood; it contains the essence of all mathematicians, possesses many charming qualities and is free from blemish.'

*Mysore* A. A. K.

G. S. DIWAN AND D. S. AGASHE—*Differential Equations*:—Karnatak Printing Press, Bombay.

The book is intended as a text book for the Indian Universities, the scope and standard being approximately the same as in the well-known text book by Murray. The authors have taken great pains to clarify the details of the methods of work, but in the opinion of the reviewer, they have overdone it. By trying to explain non-essential details and by elaborating the wordings, they have made the text book unnecessarily verbose, and a beginner who uses the book is likely to be scared away instead of being attracted into the subject. The authors say that they have attempted to cover "a number of points arising in the theory of the usual methods of solving differential equations", which are not usually dealt with in the familiar text-books. It is not clear what these points are.

C. N. S.

## ANNOUNCEMENTS AND NEWS

The following gentlemen have been admitted as members of the Indian Mathematical Society:—

1. P. C. Shah Esq., M.A. Senior Mathematics Teacher c/o Sweet Home, Madasa (Gujarat)
2. Dip Narain Lall Esq., M.A. Lecturer, G. B. B. College, Muzaffarpur.
3. T. Venkatarayudu Esq., M.A. Chief Lecturer in Mathematics, Hindu College, Masulipatam.
4. N. R. Trivedi Esq., M.A. (Bom.) B.A. (Cantab) Royal Institute of Science, Bombay
5. H. E. Peries, Esq., Secretariat, Colombo.

Prof. Maclean, Wilson College, Bombay has been elected a member of the Managing Committee of the Indian Mathematical Society in place of the late Sir Shah Mohammad Sulaiman.

## GLEANINGS

The most detailed of the symbolism definitions is the following from Prof. C. I. Lewis of Harward, the writer on symbolic logic:

"A mathematical system is any set of strings of recognisable marks in which some of the strings are taken initially and the remainder derived from these by operations performed according to rules which are independant of any meaning assigned to the marks".

I will also give Poincare's oft quoted remark "mathematicians do not study objects but the relations between objects. Matter does not engage their attention. They are interested in form alone".

TOMLINSON FORT
in Mathematics and the Sciences,
*American Mathematical Monthly*, Nov. 1940.

# SOLUTIONS TO QUESTIONS

## Question 478

(M. T. NARANIENGAR): If S, H be the foci and O the centre of the maximum inscribed ellipse of a triangle ABC, prove that

$$AS \cdot BS \cdot CS + AH \cdot BH \cdot CH = 2AO \cdot BO \cdot CO$$

*Remarks by E. H. Neville*

The question is definitely wrong as may be seen by testing it in the case of an isosceles right angled triangle. It is wrong too if the distances from the sides be substituted for the distances from the vertices. I think I know how the slip was made, but it is not worth while to print the conjecture.

## Question 1791

(P. KESAVA MENON): If through any point O three secants are drawn to meet a conic in P′, P″; Q′, Q″; R′, R″ respectively,

then
$$\frac{\Delta\, P'Q'R'}{OP' \cdot OQ' \cdot OR'} + \frac{\Delta\, P''Q''R''}{OP'' \cdot OQ'' \cdot OR''} = 0.$$

*Remarks by E. H. Neville*

The essence of this proposition is somewhat obscured by the manner in which the solution published in the *M. S.* Vol. VIII, p. 186, is concluded. In fact, as happens too often, the Cartesian scaffolding has hidden the building.

If P, Q, R are points on three concurrent lines OX, OY, OZ at distances $p$, $q$, $r$ from O then

$$2\Delta\, PQR = qr \sin YOZ + rp \sin ZOX + pq \sin XOY,$$

that is,
$$\frac{2\Delta\, PQR}{pqr} = \frac{\sin YOZ}{p} + \frac{\sin ZOX}{q} + \frac{\sin XOY}{r}.$$

This is equation (3) of the previous solution. But now all we have to remark is that from this equation it follows that if $p$, $q$, $r$ are harmonic means of three sets of distances $p_1, p_2, \cdots, p_n$; $q_1, q_2, \cdots, q_n$; $r_1, r_2, \cdots, r_n$, then $\Delta / pqr$ is the arithmetic mean of $\Delta_1 / p_1q_1r_1$, $\Delta_2 / p_2q_2r_2, \cdots, \Delta_n / p_nq_nr_n$. The sum

$$\frac{\Delta_1}{p_1q_1r_1} + \frac{\Delta_2}{p_2q_2r_2} + \cdots\cdots + \frac{\Delta_n}{p_nq_nr_n}$$

is therefore zero if and only if the area of PQR is zero, that is, if and only if P, Q, R are collinear.

Thus the result propounded expresses simply the fundamental property of the polar of O for the conic, and can be extended immediately to an algebraic curve of any degree.